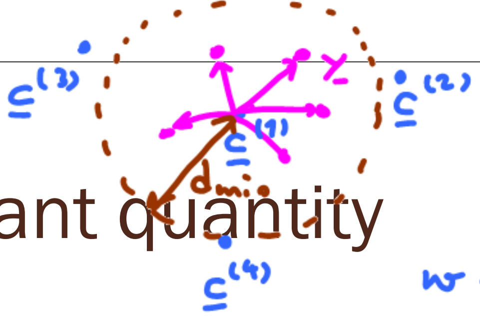


$d_{\min}$ is an important quantity

$w < d_{\min}$
$w \le d_{\min} - 1$

- To be able to detect *all* w-bit errors, we need $d_{\min} \ge w + 1$.
  - With such a code there is no way that w errors can change a valid codeword into another valid codeword.
  - When the receiver observes an illegal codeword, it can tell that a transmission error has occurred.

$d_{\min} > 2w$

- To be able to correct *all* w-bit errors, we need $d_{\min} \ge 2w + 1$. $\Rightarrow w \le \left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$
  - This way, the legal codewords are so far apart that even with w changes the original codeword is still ***closer*** than any other codeword.

$\underline{y} \oplus (-\underline{x}) = \underline{y} \oplus \underline{x}$

$d(\underline{x}, \underline{y}) = w(\underline{c})$

$\underline{x} \oplus \underline{x} = \underline{0}$

$\underline{0} \oplus \underline{x} = \underline{x}$

$\oplus$	$\underline{c}^{(1)}$	$\underline{c}^{(2)}$	$\underline{c}^{(3)}$	$\underline{c}^{(4)}$
$\underline{c}^{(1)}$	$\underline{c}^{(1)}$	$\underline{c}^{(2)}$	$\underline{c}^{(3)}$	$\underline{c}^{(4)}$
$\underline{c}^{(2)}$		$\underline{c}^{(1)}$	$\underline{c}^{(4)}$	$\underline{c}^{(3)}$
$\underline{c}^{(3)}$			$\underline{c}^{(1)}$	$\underline{c}^{(2)}$
$\underline{c}^{(4)}$				$\underline{c}^{(1)}$

Example

Consider the code

$\underline{c}^{(1)}$ $\underline{c}^{(2)}$ $\underline{c}^{(3)}$ $\underline{c}^{(4)}$

$\mathcal{C} \in \{0000000000,\ 0000011111,\ 1111100000, \text{ and } 1111111111\}$

$w(\underline{c})$: 0 5 5 10

- Is it a linear code?

  Yes.

  ① $\underline{0} \in \mathcal{C}$
  ② The sum of any pair of non-zero codewords is still a codeword.
  There are $\binom{M-1}{2}$ pairs. $|\mathcal{C}|$

- $d_{\min} = \min\limits_{\substack{\underline{c} \\ \underline{c} \ne \underline{0}}} w(\underline{c}) = 5$

- It can detect (at most) 4 errors.

  $d_{\min} - 1 = 5 - 1 = 4$

- It can correct (at most) 2 errors.

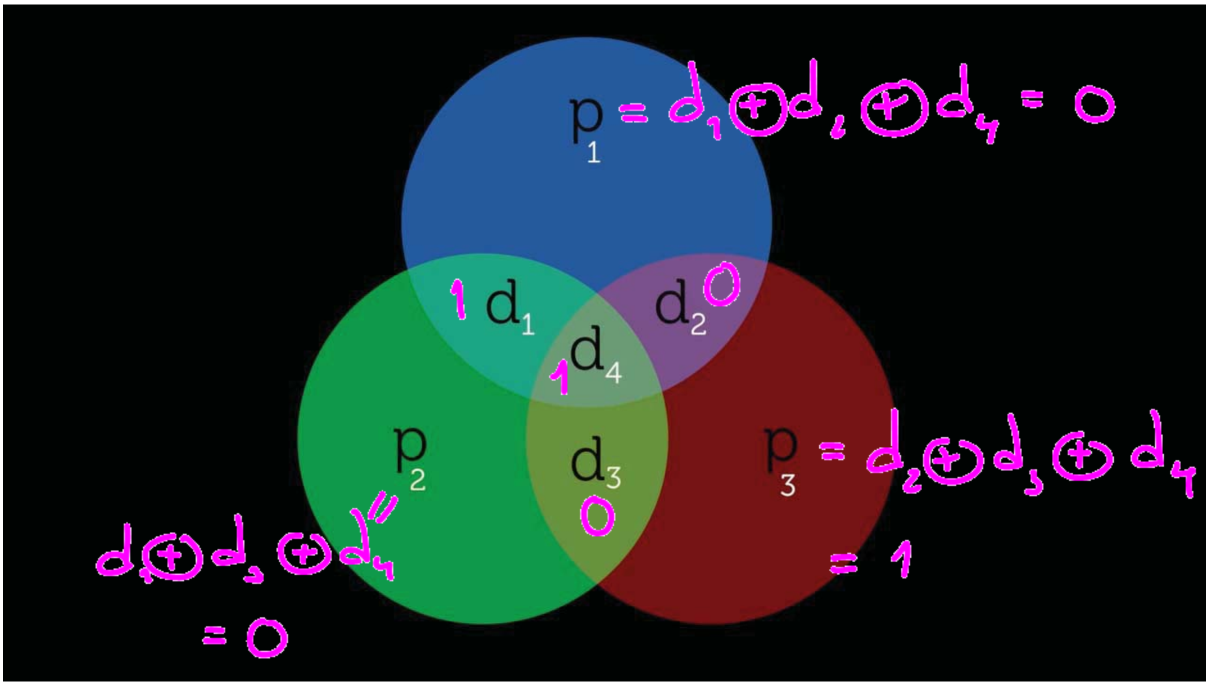
  $\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor = \left\lfloor \frac{5-1}{2} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor$

Hamming codes

- One of the earliest codes studied in coding theory.
- Named after Richard W. Hamming
  - The IEEE Richard W. **Hamming Medal**, named after him, is an award given annually by Institute of Electrical and Electronics Engineers (IEEE), for "exceptional contributions to information sciences, systems and technology“.
    - Sponsored by Qualcomm, Inc
    - Some Recipients:
      - 1988 - Richard W. Hamming
      - 1997 - Thomas M. Cover
      - 1999 - David A. Huffman
      - 2011 - Toby Berger

- The simplest of a class of (algebraic) error correcting codes that **can *correct* one error in a block of bits**

Hamming codes: Ex. 1

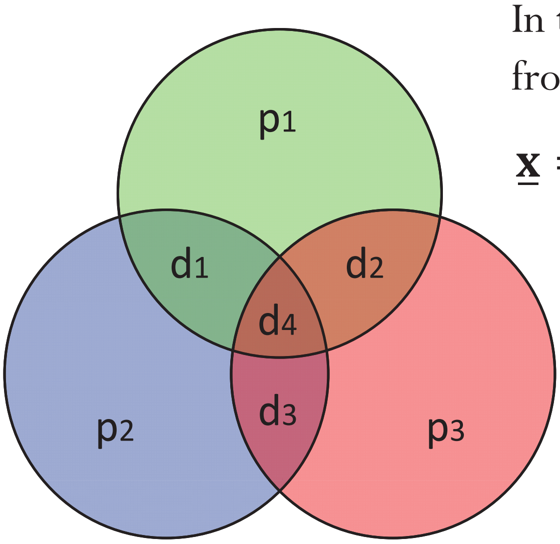


[https://www.youtube.com/watch?v=cBBTWcHkVVY]

Hamming codes: Ex. 1

n = 7
k = 4
code rate = k/n = $4/7$

This is an example of Hamming (7,4) code



In the video, the codeword is constructed from the data by

$$\underline{\mathbf{x}} = [p_1 \quad d_1 \quad p_2 \quad d_2 \quad p_3 \quad d_3 \quad d_4]$$

where

$$\begin{aligned} p_1 &= d_1 \oplus d_2 \oplus d_4 \\ p_2 &= d_1 \oplus d_3 \oplus d_4 \\ p_3 &= d_2 \oplus d_3 \oplus d_4 \end{aligned}$$

- The message bits are also referred to as the data bits or information bits.
- The non-message bits are also referred to as parity check bits, checksum bits, parity bits, or check bits.

Generator matrix: a revisit

d2

- Fact: The 1s and 0s in the j^{th} column of **G** tells which positions of the data bits are combined (⊕) to produce the j^{th} bit in the codeword.
- For the Hamming code in the previous slide,

$$\underline{\mathbf{x}} = [p_1 \quad d_1 \quad p_2 \quad d_2 \quad p_3 \quad d_3 \quad d_4] = \underline{d}\, G$$

$$= [d_1 \quad d_2 \quad d_3 \quad d_4] \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}}_{\mathbf{G}}$$

$$\begin{aligned} p_1 &= d_1 \oplus d_2 \oplus d_4 \\ p_2 &= d_1 \oplus d_3 \oplus d_4 \\ p_3 &= d_2 \oplus d_3 \oplus d_4 \end{aligned}$$

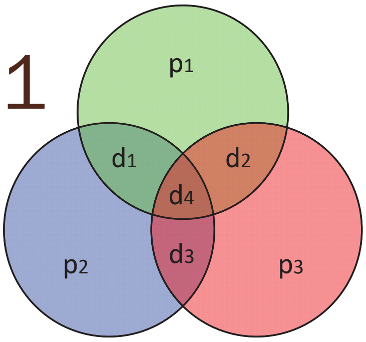
Generator matrix: a revisit

- From $\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = \sum_{j=1}^{k} b_j \underline{\mathbf{g}}_{\cancel{j}}^{(j)}$, we see that the j element of the codeword $\underline{\mathbf{x}}$ of a linear code is constructed from a linear combination of the bits in the message:

$$x_j = \sum_{i=1}^{k} b_i g_{ij} .$$

- The elements in the j^{th} column of the generator matrix become the weights for the combination.
  - Because we are working in GF(2), g_{ij} has only two values: 0 or 1.
    - When it is 1, we use b_i in the sum.
    - When it is 0, we don't use b_i in the sum.
- Conclusion: For the j^{th} column, the i^{th} element is determined from whether the i^{th} message bit is used in the sum that produces the j^{th} element of the codeword $\underline{\mathbf{x}}$.

H = Parity Check Matrix: Ex. 1



$$\begin{array}{ccccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \underline{\mathbf{x}} & = [& p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4] \end{array}$$

Structure in the codeword:

$$\begin{aligned} p_1 &= d_1 \oplus d_2 \oplus d_4 \\ p_2 &= d_1 \oplus d_3 \oplus d_4 \\ p_3 &= d_2 \oplus d_3 \oplus d_4 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} p_1 \oplus d_1 \oplus d_2 \oplus d_4 &= 0 \\ p_2 \oplus d_1 \oplus d_3 \oplus d_4 &= 0 \\ p_3 \oplus d_2 \oplus d_3 \oplus d_4 &= 0 \end{aligned}$$

At the receiver, we check whether the received vector $\underline{\mathbf{y}}$ still satisfies these conditions via computing the **syndrome vector**:

$$\underline{\mathbf{s}} = \begin{array}{c}\left[\begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \end{array}\right] \\ \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & \underline{d_1} & \underline{p_2} & d_2 & p_3 & \underline{d_3} & \underline{d_4} \end{array}\end{array} \overset{H^T}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underline{0}?$$

Parity Check Matrix: Ex 1

- Intuitively, the **parity check matrix** $\mathbf{H}$, as the name suggests, tells which bits in the observed vector $\underline{\mathbf{y}}$ are used to “check” for validity of $\underline{\mathbf{y}}$.
- The number of rows is the same as the number of conditions to check (which is the same as the number of parity check bits).
- For each row, a one indicates that the bits (including the bits in the parity positions) are used in the validity check calculation.

Structure in the codeword:

$$\begin{aligned} p_1 \oplus d_1 \oplus d_2 \oplus d_4 &= 0 \\ p_2 \oplus d_1 \oplus d_3 \oplus d_4 &= 0 \\ p_3 \oplus d_2 \oplus d_3 \oplus d_4 &= 0 \end{aligned} \quad \Longleftrightarrow \quad \mathbf{H} = \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

Parity Check Matrix: Ex 1

Relationship between $\mathbf{G}$ and $\mathbf{H}$.

$$\mathbf{G} = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \quad \Longleftrightarrow \quad \mathbf{H} = \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

Parity Check Matrix: Ex 1

Relationship between **G** and **H**.

$$\mathbf{G} = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \Longleftrightarrow \mathbf{H} = \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

Parity Check Matrix: Ex 1

Relationship between **G** and **H**.

$$\mathbf{G} = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \Longleftrightarrow \mathbf{H} = \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

(columns of) identity matrix in the data positions

(columns of) identity matrix in the parity check positions

Parity Check Matrix: Ex 1

Relationship between **G** and **H**.

$$\mathbf{G} = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \Leftrightarrow \mathbf{H} = \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ p_1 & d_1 & p_2 & d_2 & p_3 & d_3 & d_4 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

Parity Check Matrix

Key property:

$$\boxed{\mathbf{G}\mathbf{H}^T = \mathbf{0}_{k\times(n-k)}}$$

Proof:

- When there is no error $(\underline{\mathbf{e}} = \underline{\mathbf{0}})$, the syndrome vector calculation should give $\underline{\mathbf{s}} = \underline{\mathbf{0}}$.
- By definition,

  $$\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T = (\underline{\mathbf{x}} \oplus \underline{\mathbf{e}})\mathbf{H}^T = \underline{\mathbf{x}}\mathbf{H}^T \oplus \underline{\mathbf{e}}\mathbf{H}^T = \underline{\mathbf{b}}\mathbf{G}\mathbf{H}^T \oplus \underline{\mathbf{e}}\mathbf{H}^T.$$

- Therefore, when $\underline{\mathbf{e}} = \underline{\mathbf{0}}$, we have $\underline{\mathbf{s}} = \underline{\mathbf{b}}\mathbf{G}\mathbf{H}^T$.
- To have $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ for any $\underline{\mathbf{b}}$, we must have $\mathbf{G}\mathbf{H}^T = \underline{\mathbf{0}}$.

Systematic Encoding

- Code constructed with distinct information bits and check bits in each codeword are called **systematic codes**.
  - **Message bits are "visible" in the codeword**.
- Popular forms of **G**:

$$\mathbf{G} = \left[\mathbf{P}_{k\times(n-k)} \;\vdots\; \mathbf{I}_k\right]$$

$$\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = \left[b_1 \quad b_2 \quad \cdots \quad b_k\right]\left[\mathbf{P}_{k\times(n-k)} \;\vdots\; \mathbf{I}_k\right]$$

$$= \left[x_1 \quad x_2 \quad \cdots \quad x_{n-k} \;\vdots\; \underset{x_{n-k+1}}{b_1} \quad \underset{x_{n-k+2}}{b_2} \quad \cdots \quad \underset{x_n}{b_k}\right]$$

$$\mathbf{G} = \left[\mathbf{I}_k \;\vdots\; \mathbf{P}_{k\times(n-k)}\right]$$

$$\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = \left[b_1 \quad b_2 \quad \cdots \quad b_k\right]\left[\mathbf{I}_k \;\vdots\; \mathbf{P}_{k\times(n-k)}\right]$$

$$= \left[\underset{x_1}{b_1} \quad \underset{x_2}{b_2} \quad \cdots \quad \underset{x_k}{b_k} \;\vdots\; x_{k+1} \quad x_{k+2} \quad \cdots \quad x_n\right]$$

Parity check matrix

- For the generators matrices we discussed in the previous slide, the corresponding **parity check matrix** can be found easily:

$$\mathbf{G} = \left[\mathbf{P}_{k\times(n-k)} \;\vdots\; \mathbf{I}_k\right] \Longrightarrow \mathbf{H} = \left[\mathbf{I}_{n-k} \;\vdots\; -\mathbf{P}^T\right]$$

Check: $\mathbf{G}\mathbf{H}^T = \left[\mathbf{P} \;\vdots\; \mathbf{I}\right]\begin{bmatrix}\mathbf{I} \\ \text{-}\mathbf{P}\end{bmatrix} = \mathbf{P} \oplus \left(-\mathbf{P}\right) = \mathbf{0}_{k\times(n-k)}$

$$\mathbf{G} = \left[\mathbf{I}_k \;\vdots\; \mathbf{P}_{k\times(n-k)}\right] \Longrightarrow \mathbf{H} = \left[-\mathbf{P}^T \;\vdots\; \mathbf{I}_{n-k}\right]$$

Hamming codes: Ex. 2

- Systematic (7,4) Hamming Codes

$$\mathbf{G} = \left[\overbrace{\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}}^{P} \middle| \overbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}^{I_4} \right]$$

$$\mathbf{H} = \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \middle| \overbrace{\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{matrix}}^{P^T} \right]$$

Hamming codes

Now, we will gives a general recipe for constructing Hamming codes.

Parameters:

- $\overset{\gamma,\, r}{m} = n - k =$ number of parity bits   2   3   4 ....
- $n = 2^m - 1 \in \{3,7,15,31,63,127, \dots\}$ 3   7   15 ..
- $k = n - m = 2^m - m - 1$ 1   4   11 ...

It can be shown that, for Hamming codes,

- $d_{\min} = 3.$
- Error correcting capability: $t = 1$ $\leftarrow \left\lfloor \frac{d_{min} - 1}{2} \right\rfloor$

Construction of Hamming Codes

Ex. $m = 2$

$2^m - 1 = 2^2 - 1 = 3$

- Start with m.

1. Parity check matrix $\mathbf{H}$:
   - Construct a matrix whose columns consist of *all* nonzero binary m-tuples.
   - The ordering of the columns is arbitrary. However, next step is easy when the columns are arranged so that $\mathbf{H} = \left[\mathbf{I}_m \;\vdots\; \mathbf{P}\right]$.

$$\mathbf{H} = \left[\begin{array}{c|c|c|c} \cancel{0} & 0 & 1 & 1 \\ \cancel{0} & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

2. Generator matrix $\mathbf{G}$:
   - When $\mathbf{H} = \left[\mathbf{I}_m \;\vdots\; \mathbf{P}\right]$, we have $\mathbf{G} = \left[-\mathbf{P}^T \;\vdots\; \mathbf{I}_k\right] = \left[\mathbf{P}^T \;\vdots\; \mathbf{I}_k\right]$.

$$G = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

same as repetition code

$m = 3$

Hamming codes: Ex. 2

- Systematic (7,4) Hamming Codes

$$\mathbf{H} = \left[\begin{array}{ccc:cccc} & & & & -\mathbf{P}^T & & \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

- Columns are all possible 3-bit vectors
- We arrange the columns so that $\mathbf{I}_3$ is on the left to make the code systematic. (One can also put $\mathbf{I}_3$ on the right.)

$$\mathbf{G} = \left[\begin{array}{ccc:cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ & \mathbf{P} & & & & & \end{array}\right]$$

- Note that the size of the identity matrices in $\mathbf{G}$ and $\mathbf{H}$ are not the same.

Minimum Distance Decoding

- At the decoder, suppose we want to use minimum distance decoding, then
  - The decoder needs to have the list of all the possible codewords so that it can compare their distances to the received vector $\underline{\mathbf{y}}$.
  - There are 2^k codewords each having n bits.
    Therefore, saving these takes $2^k \times n$ bits.
  - Also, we will need to perform the comparison 2^k times.
- Alternatively, we can utilize the syndrome vector (which is computed from the parity-check matrix).
  - The syndrome vector is computed from the parity-check matrix $\mathbf{H}$.
  - Therefore, saving $\mathbf{H}$ takes $(n-k) \times n$ bits.

Minimum Distance Decoding

- Observe that

$$d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = w(\underline{\mathbf{x}} \oplus \underline{\mathbf{y}}) = w(\underline{\mathbf{e}})$$

- Therefore, minimizing the distance is the same as minimizing the weight of the error pattern.
- New goal:
  - find the decoded error pattern $\underline{\hat{\mathbf{e}}}$ with the minimum weight
  - then, the decoded codeword is $\underline{\hat{\mathbf{x}}} = \underline{\mathbf{y}} \oplus \underline{\hat{\mathbf{e}}}$
- Once we know $\underline{\hat{\mathbf{x}}}$ we can directly extract the message part from the decoded codeword if we are using systematic code.
- For example, consider

$$\mathbf{G} = \left[\begin{array}{ccc:cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Suppose $\underline{\hat{\mathbf{x}}} = 1011010$, then we know that the decoded message is $\underline{\hat{\mathbf{b}}} = 1010$.

Properties of Syndrome Vector

- From $\mathbf{GH}^T = \mathbf{0}$, we have

$$\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T = (\underline{\mathbf{x}} \oplus \underline{\mathbf{e}})\mathbf{H}^T = (\underline{\mathbf{b}}\mathbf{G} \oplus \underline{\mathbf{e}})\mathbf{H}^T \overset{\overset{GH^T = 0}{\downarrow}}{=} \underline{\mathbf{e}}\mathbf{H}^T$$

- Thinking of $\mathbf{H}$ as a matrix with many columns inside,

$$\mathbf{H} = \begin{bmatrix} \underline{\mathbf{h}}_1 \\ \boxed{\quad \underline{\mathbf{h}}_2 \quad} \\ \vdots \\ \underline{\mathbf{h}}_{n-k} \end{bmatrix}_{(n-k)\times n} = \begin{bmatrix} \underline{\mathbf{v}}_1^T & \boxed{\underline{\mathbf{v}}_2^T} & \cdots & \underline{\mathbf{v}}_n^T \end{bmatrix}$$

$$\underline{\mathbf{s}} = \underline{\mathbf{e}}\mathbf{H}^T = \sum_{j=1}^{n} e_j \underline{\mathbf{v}}_j$$

- Therefore, $\underline{\mathbf{s}}$ is a linear combination of the columns of $\mathbf{H}$.

Hamming Codes: Ex. 2

$$\mathbf{H} = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\underline{\mathbf{s}} = \underline{\mathbf{e}}\mathbf{H}^T = \underbrace{\sum_{j=1}^{n} e_j \underline{\mathbf{v}}_j}_{\text{Linear combination of the columns of } \mathbf{H}}$$

Note that for an error pattern with a single one in the j^{th} coordinate position, the syndrome $\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T$ is the same as the j^{th} column of $\mathbf{H}$.

Error pattern $\underline{\mathbf{e}}$	Syndrome = $\underline{\mathbf{e}}\mathbf{H}^T$
(0,0,0,0,0,0,0)	(0,0,0)
(0,0,0,0,0,0,1)	(1,1,1)
(0,0,0,0,0,1,0)	(1,1,0)
(0,0,0,0,1,0,0)	(1,0,1)
(0,0,0,1,0,0,0)	(0,1,1)
(0,0,1,0,0,0,0)	(0,0,1)
(0,1,0,0,0,0,0)	(0,1,0)
(1,0,0,0,0,0,0)	(1,0,0)

(0 1 1 0 0 0 0) $(0\,1\,0) \oplus (0\,0\,1) = 0\,1\,1$

Properties of Syndrome Vector

- We will assume that the columns of **H** are nonzero and distinct.
  - This is automatically satisfied for Hamming codes constructed from our recipe.
- When $\underline{\mathbf{e}} = \underline{\mathbf{0}}$, we have $\underline{\mathbf{s}} = \underline{\mathbf{0}}$.
  - When $\underline{\mathbf{s}} = \underline{\mathbf{0}}$, we can conclude that $\hat{\underline{\mathbf{e}}} = \underline{\mathbf{0}}$.
    - There can also be $\underline{\mathbf{e}} \neq \underline{\mathbf{0}}$ that gives $\underline{\mathbf{s}} = \underline{\mathbf{0}}$.
      - For example, any nonzero $\tilde{\underline{\mathbf{e}}} \in \mathcal{C}$, will also give $\underline{\mathbf{s}} = \underline{\mathbf{0}}$.
      - However, they have larger weight than $\underline{\mathbf{e}} = \underline{\mathbf{0}}$.
    - The decoded codeword is the same as the received vector.
- When, $e_i = \begin{cases} 0, & i = j, \\ 1, & i \neq j, \end{cases}$ (a pattern with a single one in the j^{th} position) we have $\underline{\mathbf{s}} = \underline{\mathbf{v}}_j$ = the j^{th} column of **H**.
  - When $\underline{\mathbf{s}}$ = the j^{th} column of **H**, we can conclude that $\hat{e}_i = \begin{cases} 0, & i = j, \\ 1, & i \neq j, \end{cases}$
    - There can also be other $\underline{\mathbf{e}}$ that give $\underline{\mathbf{s}} = \underline{\mathbf{v}}_j$. However, their weights
      - can not be 0 (because, if so, we would have $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ but the columns of **H** are nonzero)
      - nor 1 (because the columns of **H** are distinct).
    - We flip the j^{th} bit of the received vector to get the decoded codeword.

~~Hamming Codes:~~ Decoding Algorithm

- For general linear codes, the two cases discussed on the previous slide may not cover every cases.
- For Hamming codes, because the columns are constructed from all possible non-zero *m*-tuples, the syndrome vectors must fall into one of the two cases considered.
- ~~Hamming Codes:~~ Decoding Recipe
  - Compute the syndrome $\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T$ for the received vector.
  - Case 1: If $\underline{\mathbf{s}} = \underline{\mathbf{0}}$, set $\hat{\underline{\mathbf{x}}} = \underline{\mathbf{y}}$.
  - Case 2: If $\underline{\mathbf{s}} \neq \underline{\mathbf{0}}$,
    - Determine the position j of the column of **H** that is the transposition of the syndrome.
    - set $\hat{\underline{\mathbf{x}}} = \underline{\mathbf{y}}$ but with the j^{th} bit complemented.

Properties of Syndrome Vector

- We will assume that the columns of **H** are nonzero and distinct.
  - This is automatically satisfied for Hamming codes constructed from our recipe.
- Case 1: When $\underline{\mathbf{e}} = \underline{\mathbf{0}}$, we have $\underline{\mathbf{s}} = \underline{\mathbf{0}}$.
  - When $\underline{\mathbf{s}} = \underline{\mathbf{0}}$, we can conclude that $\underline{\hat{\mathbf{e}}} = \underline{\mathbf{0}}$.
    - There can also be $\underline{\mathbf{e}} \neq \underline{\mathbf{0}}$ that gives $\underline{\mathbf{s}} = \underline{\mathbf{0}}$.
      - For example, any nonzero $\underline{\tilde{\mathbf{e}}} \in \mathcal{C}$, will also give $\underline{\mathbf{s}} = \underline{\mathbf{0}}$.
      - However, they have larger weight than $\underline{\mathbf{e}} = \underline{\mathbf{0}}$.
    - The decoded codeword is the same as the received vector.
- Case 2: When, $e_i = \begin{cases} 0, & i = j, \\ 1, & i \neq j, \end{cases}$ (a pattern with a single one in the j^{th} position) we have $\underline{\mathbf{s}} = \underline{\mathbf{v}}_j$ = the j^{th} column of **H**.
  - When $\underline{\mathbf{s}}$ = the j^{th} column of **H**, we can conclude that $\hat{e}_i = \begin{cases} 0, & i = j, \\ 1, & i \neq j, \end{cases}$
    - There can also be other $\underline{\mathbf{e}}$ that give $\underline{\mathbf{s}} = \underline{\mathbf{v}}_j$. However, their weights
      - can not be 0 (because, if so, we would have $\underline{\mathbf{s}} = \underline{\mathbf{0}}$ but the columns of **H** are nonzero)
      - nor 1 (because the columns of **H** are distinct).
    - We flip the j^{th} bit of the received vector to get the decoded codeword.

(Modified)

Decoding Algorithm

- Assumption: the columns of **H** are nonzero and distinct.
- Compute the **syndrome** $\underline{\mathbf{s}} = \underline{\mathbf{y}}\mathbf{H}^T$ for the received vector.
- Case 1: If $\underline{\mathbf{s}} = \underline{\mathbf{0}}$, set $\underline{\hat{\mathbf{x}}} = \underline{\mathbf{y}}$.
- Case 2: If $\underline{\mathbf{s}} \neq \underline{\mathbf{0}}$,
  - determine the position j of the column of **H** that is the same as (the transposition) of the syndrome,
  - set $\underline{\hat{\mathbf{x}}} = \underline{\mathbf{y}}$ but with the j^{th} bit complemented.
- For Hamming codes, because the columns are constructed from all possible non-zero m-tuples, the syndrome vectors must fall into one of the two cases considered.
- For general linear block codes, the two cases above may not cover every cases.

(Modified)

Hamming Codes: Ex. 1

- Consider the Hamming code with

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \Leftrightarrow \mathbf{H} = \begin{bmatrix} 1 & \boxed{1} & 0 & 1 & 0 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 & 0 & 1 & 1 \\ 0 & \boxed{0} & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

- Suppose we observe $\underline{\mathbf{y}} = [0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1]$ at the receiver. Find the decoded codeword and the decoded message.

$\underline{s} = \underline{y} H^T = \underline{(1\ 1\ 0)}$ → same as the 2nd column of H

$$\hat{\underline{x}} = [0\ \overset{0}{\underline{\cancel{1}}}\ 0\ \underline{1}\ 1\ \underline{1}\ \underline{1}]$$

$$\left.\begin{matrix} \hat{\underline{b}} \\ \hat{\underline{d}} \\ \hat{\underline{m}} \end{matrix}\right\} = [0\ 1\ 1\ 1]$$

Interleaving

- Conventional error-control methods such as parity checking are designed for errors that are isolated or statistically independent events.
- Some errors occur in bursts that span several successive bits.
  - These errors tend to group together in bursts.
    Thus, they are no longer independent.
  - Examples
    - impulse noise produced by lightning and switching transients
    - fading in wireless systems
    - channel with memory
- Such multiple errors wreak havoc on the performance of conventional codes and must be combated by special techniques.
- One solution is to spread out the transmitted codewords.
- We consider a type of interleaving called **block interleaving**.

Interleave as a verb

- To interleave = to combine different things so that parts of one thing are put between parts of another thing
- Ex. To interleave two books together:

Interleaving: Example

Consider a sequence of m blocks of coded data:

$$\left(x_1^{(1)}x_2^{(1)}\cdots x_n^{(1)}\right)\left(x_1^{(2)}x_2^{(2)}\cdots x_n^{(2)}\right)\cdots\left(x_1^{(\ell)}x_2^{(\ell)}\cdots x_n^{(\ell)}\right)$$

$$\downarrow$$

$$\begin{matrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_n^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(\ell)} & x_2^{(\ell)} & \cdots & x_n^{(\ell)} \end{matrix}$$

$$\downarrow$$

$$\left(x_1^{(1)}x_1^{(2)}\cdots x_1^{(\ell)}\right)\left(x_2^{(1)}x_2^{(2)}\cdots x_2^{(\ell)}\right)\cdots\left(x_n^{(1)}x_n^{(2)}\cdots x_n^{(\ell)}\right)$$

- Arrange these blocks as rows of a table.
- Normally, we get the bit sequence simply by reading the table by rows.
- With interleaving (by an interleaver), transmission is accomplished by reading out of this table by columns.
- Here, ℓ blocks each of length n are interleaved to form a sequence of length ℓn.

The received symbols must be deinterleaved (by a deinterleaver) prior to decoding.

Interleaving: Advantage

- Consider the case of a system that can only correct single errors.
- If an error burst happens to the original bit sequence, the system would be overwhelmed and unable to correct the problem.

original bit sequence $\left(x_1^{(1)}x_2^{(1)}\cdots x_n^{(1)}\right)\left(x_1^{(2)}x_2^{(2)}\cdots x_n^{(2)}\right)\cdots\left(x_1^{(\ell)}x_2^{(\ell)}\cdots x_n^{(\ell)}\right)$

interleaved transmission $\left(x_1^{(1)}x_1^{(2)}\cdots x_1^{(\ell)}\right)\left(x_2^{(1)}x_2^{(2)}\cdots x_2^{(\ell)}\right)\cdots\left(x_n^{(1)}x_n^{(2)}\cdots x_n^{(\ell)}\right)$

- However, in the interleaved transmission,
  - successive bits which come from ***different*** original blocks have been corrupted
  - when received, the bit sequence is reordered to its original form and then the FEC can correct the faulty bits
  - Therefore, single error-correction system is able to fix several errors.

Interleaving: Advantage

- If a burst of errors affects at most ℓ consecutive bits, then each original block will have at most one error.
- If a burst of errors affects at most $r\ell$ consecutive bits (assume $r < n$), then each original block will have at most r errors.
- Assume that there are no other errors in the transmitted stream of ℓn bits.
  - A single error-correcting code can be used to correct a single burst spanning upto ℓ symbols.
  - A double error-correcting code can be used to correct a single burst spanning upto 2ℓ symbols.